Rethinking Topology (or a Personal Topologodicy)

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When I was originally introduced to topology, I simply accepted most of its properties as generalizations of $\mathbb{R}^n$. I didn't give it any serious thought until about a month ago when I read an excellent thread on math overflow about it. Since then, its been one of the things I often find myself thinking about when I'm trying to fall asleep. Given the amount of thought I've put into it, and the fact that I feel I should be answer questions like this about topology, given that it's one of the areas of math I spend a lot of time on, I thought I'd write up my thoughts. They lent themselves well to being written in the form of an introduction to topology, so that's what I did.

(After finishing this essay I decided to reread the MO thread. The first comment – not answer, a comment – mentions the Kuratowski closure axioms and closure axioms sounded like one might call what I came up with. Sure enough, they're the exact same, down to the ordering. Are all attempts to make mathematical contribution's this frustrating? I'm posting this because of the amount of work I put in, but there's nothing new here.)

Consider 1 with respect to $[0, 1)$. It isn't part of the set, but in a sort of intuitive sense it almost is. And knowing which points are 'almost in' a set gives us lots of information, for example notions of boundaries and connectedness. Topology is based on us formalizing this notion of 'almost in' and once we formalize it, we can consider non-standard notions of being 'almost in' or apply these ideas to spaces that we don't typically associate them with.

We call points that are in or 'almost in' a set the 'adherant points' of a set. We call the set of adherant points of a set the closure of that set; the closure of a set s is denoted as $\mathrm{cl}(s)$ or $\overline{s}$. From the closure, we can define lots of other niceties: the interior of a set, $\mathrm{int}(S) = \mathrm{cl}(S^C)^C$, or the boundary, $\mathrm{bd}(S) = \mathrm{cl}(S) \cap \mathrm{cl}(S^C)$. Notice that we can also define closure in terms of either of these (like so: $\mathrm{cl}(S) = \mathrm{int}(S^C)^C = S \cup \mathrm{bd}(S)$) and, as such, they all contain the same information.

We call a set which is its own closure is called a closed set. We call the complement of a closed set an open set. One can think of a closed set as one which contains its own boundary, and an open set as one which does not contain its own boundary.

At this point, closure could be any map from the power set (set of all possible subsets) of our space to itself. Obviously, many of these are nonsensical, for example $\mathrm{cl}(S) = \emptyset$ would mean that points in a set aren't adherant points of that set! So we're going to put four restrictions on closure – they're very important because they result in restrictions on our idea of a generic topological space and they're the only restrictions we're going to put on it.

- **Restriction 1**: This one has been stated informally when we introduced the notion of a closure: points of a set are adherant points of that set, or $S \subseteq \mathrm{cl}(S)$.
- **Restriction 2**: The closure of a set is closed, or $\mathrm{cl}(cl(S)) = \mathrm{cl}(S)$. This can be thought of as meaning that adding the boundary of a set to itself doesn't create a new boundary.
- **Restriction 3**: The closure of the union of two sets is the union of the closures. This can be thought of as meaning that the union of two sets have a boundary that is the subset of their original boundaries.
- **Restriction 4**: The empty set has no adherant points, or $\mathrm{cl}(\emptyset) = \emptyset$. Alternatively, $\emptyset$ is closed.

Corollary of restriction 3: The closure of a big set contains the closure of a smaller set, or $A \subseteq B \rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. **Proof**: $\mathrm{cl}(B) = \mathrm{cl}((B \cap A) \cup (B \setminus A)) = \mathrm{cl}(B \cap A) \cup \mathrm{cl}(B \setminus A) = \mathrm{cl}(A) \cup \mathrm{cl}(B \setminus A) \supseteq \mathrm{cl}(A)$. ■

These restrictions are equivalent to the following restrictions on interior:

- **Restriction 1**: The interior of a set is a subset of the original set, or $\mathrm{int}(S) \subseteq S$.
- **Restriction 2**: The interior of a set is open, or $\mathrm{int}(int(S)) = \mathrm{int}(S)$
- **Restriction 3**: The interior of the intersections of two sets is the intersection of the interiors, or $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap int(B)$.
- **Restriction 4**: The interior of the space is the space, or $\mathrm{int}(X) = X$.

Restrictions 2 and 4 easily map to restrictions on boundary as well. Restriction 2 is equivalent to boundaries being their own boundary while restriction 4 is equivalent to the empty set (or, equivalently, the entire space) having no boundary.

These restrictions are enough to allow us to define the closure of a set in terms of closed sets as the minimal closed set around it. **Proof**: Let F be an arbitrary closed super set of S. By (corollary of 3), $\mathrm{cl}(S) \subseteq \mathrm{cl}(F) = F$. Thus $\mathrm{cl}(S)$ is a closed (by (2)) super set (by (1)) of S that is a subset or equal to any closed super set of S. Therefore a minimal closed super set of S exists and is $\mathrm{cl}(S)$. ■

We can also define $\mathrm{cl}(S)$ in terms of open sets. Let us define $\mathrm{cl}'(S)$ as the set of points x such that all open sets containing x have a non-empty intersection with S. I claim that $\mathrm{cl}(S) = \mathrm{cl}'(S)$.

Proof: It suffices to show that $x \in \mathrm{cl}(S)$ iff $x \in \mathrm{cl}'(S)$. We break this into two separate claims, implication in each direction.

$x \in \mathrm{cl}(S) \to x \in \mathrm{cl}'(S)$: Note that this is equivalent to $x \notin \mathrm{cl}'(S) \to x \notin \mathrm{cl}(S)$, which is fairly easy to see. If not every open set containing x has a non-empty intersection with S, then there is an open set containing it with an empty intersection with S; its complement is a closed super set of S that does not contain x, contradicting x belonging to the minimal closed super set of S.

$x \in \mathrm{cl}'(S) \to x \in \mathrm{cl}(S)$: Again, we use proof by contradiction (ie. prove $x \notin \mathrm{cl}(S) \to x \notin \mathrm{cl}'(S)$). If x does not belong to the minimal closed super set of S, there is a closed super of set of S not containing it; its complement is an open set containing x which does not intersect S, and thus $x \notin \mathrm{cl}'(S)$. ■

While the definition in terms of closed sets is simpler, the definition in terms of open sets reveals a lot more about what the closure actually is, and is a lot more powerful to use in proofs.

If we were working in a metric space (that is, a space with a notion of distance between its elements), one could naturally define the notion of x being in the closure of S as there being a point in S that is arbitrarily close to x, that is $x \in \mathrm{cl}(S)$ iff $(\forall \epsilon > 0)(\exists y \in S)(d(x, y) < \epsilon)$. Requiring that there is an element of S in every open set containing x, that is $(\forall u, u\text{open}, x \in u)(\exists y \in S)(y \in u)$ or alternatively $(\forall u, u\text{open}, x \in u)(u \cap s \neq \emptyset)$, is the topological equivalent of this.

(This collection of open sets is an example of a filter, an important tool in topology. Checking for every open set may seem tedious, but you can usually find a much smaller number of sets that it suffices to check because all other open sets are super sets of them. For example, in $\mathbb{R}^{\mathrm{n}}$, it suffices to check the balls of radius 2^{-j} about a point, where $j \in \mathbb{N}$. This is an example of what we will later call a countable local basis; spaces with a countable local basis around every point are called first countable.)

Since open sets containing an element x are an idea we will be using a lot, we often call them neighborhoods of x.

I previously claimed that the definition of a closure in terms of open sets was very powerful. I will now demonstrate this by proving some important properties of closed sets through it.

It is easy to see, from our original restrictions, that the intersection of a collection of closed sets $\mathscr{C}$ is closed: $\mathrm{cl}(\bigcap \mathscr{C}) \supseteq \bigcap \mathscr{C}$ by (restriction 1) and $(\forall C \in \mathscr{C})(cl(\bigcap \mathscr{C}) \subset \mathrm{cl}(C) = C)$ (by restriction 2) which implies $\mathrm{cl}(\bigcap \mathscr{C}) \subseteq \bigcap \mathscr{C}$. Therefore, $\mathrm{cl}(\bigcap \mathscr{C}) = \bigcap \mathscr{C}$ is closed. ■

But a proof that offers deeper insight into the nature of closed sets is: Suppose $x \in \mathrm{cl}(\bigcap \mathscr{C})$. Then for all neighborhoods of x, there is an element in the neighborhood which is in all elements of $\mathscr{C}$. Then, for any particular element C of $\mathscr{C}$, $x \in \mathrm{cl}(C)$ since all neighborhoods of it contain some element of C. Since $\mathscr{C}$ is a collection of closed sets, C is closed and $\mathrm{cl}(C) = C$. Therefore $x \in C$, for all $C \in \mathscr{C}$. Therefore $x \in \bigcap \mathscr{C}$. Therefore $\mathrm{cl}(\bigcap \mathscr{C}) = \bigcap \mathscr{C}$. Therefore, $\bigcap \mathscr{C}$ is closed. ■

While it is obvious by finite induction on restriction 3 that the union of finitely many closed sets is closed, the question is left hanging whether this is true for a general collection of closed sets $\mathscr{C}$. Our definition in terms of open sets answers this question: suppose $x \in \mathrm{cl}(\bigcup \mathscr{C})$ and let $C_1, C_2, C_3 ... \in \mathscr{C}$. It is possible that $x \in \mathrm{cl}(\bigcup \mathscr{C})$ simply because a couple of its neighborhoods intersect C_1, a few others C_2, and so on, but don't all intersect any element of $\mathscr{C}$. Thus, there are a few ways to resolve this. Obviously, we could just give up on cases where $\mathscr{C}$ is finite, but the restriction of being locally finite (ie. every point has an open set that intersects only finitely many elements of $\mathscr{C}$) on points not in any element of $\mathscr{C}$ would also suffice.

Proof: First note that the fact that the union of finitely many closed sets is closed is equivalent, by De Morgan's Laws, to the finite intersection of opens sets being open.

Let $x \in \mathrm{cl}(\bigcup \mathscr{C})$. We wish to show that $x \in \bigcup \mathscr{C}$, thereby proving that $\mathrm{cl}(\bigcup \mathscr{C}) = \bigcup \mathscr{C}$ and thus that $\bigcup \mathscr{C}$ is closed. If x is in an element of $\mathscr{C}$, then $x \in \bigcup \mathscr{C}$ and we're done, so suppose not. Then by local finiteness, there is neighborhood of x that intersects only finitely many elements of $\mathscr{C}$. Let us call these $C_1, C_2, C_3 ... C_i$. Suppose C_j doesn't intersect all neighborhoods of x; let U be an neighborhood that it doesn't intersect, then given V, an open set that intersects C_j, we can find an open (as proven in the preceding paragraph) subset of V that is not intersected by C_j, $U \cap V$. Since C_j doesn't intersect it, another element does and thus we may safely remove C_j from our finite collection and still have it intersect all neighborhoods of x. We repeat this until we only have one set (which must, then, intersect all neighborhoods of x) or all the sets remaining intersect all neighborhoods. Either way, x is an adherant point of the remaining sets, of which there is at least one, and since they're closed, in them. Thus, $x \in \bigcup \mathscr{C}$ and $\bigcup \mathscr{C}$ is closed. ■

Two particular sets we should consider are the empty set, $\emptyset$, and the space, X. By restriction 4, $\mathrm{cl}(\emptyset) = \emptyset$ and thus $\emptyset$ is closed. On the other hand, $\mathrm{cl}(X) \supseteq X$ and it isn't possible for a set to be a super set of X and still be in our space, X is closed.. Since $\emptyset$ and X are complements and are both closed, they are both open.

Thus $\emptyset$ and X are examples of sets that are both closed and open. It may seem unintuitive that a set can be both closed and open, but there is actually quite a simple reason for it. A set being close means it contains its boundary, as set being open means that it doesn't contain its boundary. Thus, a set being clopen (that is, both closed an open) means that it has no boundary and both contains and doesn't contain it.

We are now prepared to return the question of what a topology is. At the beginning of this essay, I said that in topology we formalize the notion being 'almost in', and we did in the form of closure. But we also found four other ways of conveying the same information: interior, boundary, closed sets and open sets. While any of these can describe the topology of a space, it would be nice to agree on a standard way.

Closure, interior and boundary are ruled out for several reasons. The first is that mathematicians are generally minimalists and a function from the power set of X to itself is a far more complicated than a subset of the power set. Another reason is that describing these functions in most topological spaces is very awkward without first defining open or closed sets. This leaves us with a choice between open sets and closed sets.

While closed sets may seem, on first glance, more elegant, I hope that this essay has demonstrated the utility of open sets. This utility, combined with history, result in us using open sets instead of closed ones. (It is also worth noting that a great deal of the apparent simplicity of open sets relative to closed sets is that we focused on closure instead of interior in this essay.)

Definition: A topology is the collection of open sets on a space.

The restrictions we put on closure map to restrictions on the topology:

- **Restriction 1**: $\emptyset$ and X are in the topology.
- **Restriction 2**: The finite intersection of sets in the topology is in the topology.
- **Restriction 3**: The arbitrary union of sets in the topology is in the topology.

The normal definition of a topology on X is a collection of sets meeting these restrictions. They are equivalent to our previous restrictions.

Proof: We previously proved all of these to be consequence of our original restrictions on closure. We will now prove that our restrictions on closure are consequences of these, thereby demonstrating that they are equivalent.

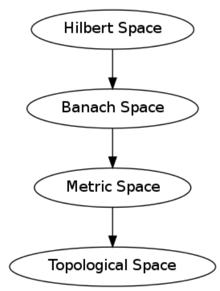
- Restriction 1 on closure, that $\text{cl}(S) \supseteq S$, is a consequence of the fact that the minimal closed super set of a set is a super set.
- Restriction 2 on closure, that $\text{cl}(cl(S)) = \text{cl}(S)$ or that the closure of a closed is closed, is a result of a closed set being the minimal closed super set of itself.
- Restriction 3 on closure, that $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$, is a consequence of applying De Morgan's Laws to restriction 2 on a topology.
- Restriction 4 on closure, that $\emptyset$ is closed follows from X being open.

Therefore, the restrictions on a topology are equivalent to the restrictions I put on closure. ■

At this point, I think it is fairly clear that these are reasonable restrictions. The question that remains is whether we should put other restrictions on topologies. If you are new to topology, you probably have lots of other ideas as to how open and closed sets should behave. For example, that given two points you can find an open set around one not containing the other. While this is true in many spaces, and quite likely true in most spaces you've seen, it is not true in all spaces (look up Hausdorff distance). However, since it is quite common, we still study these spaces, calling them T_0 topological spaces. T_0 is an example of a 'separation axiom', a further restriction we place on topological spaces. There are a number of separation axioms, most notably $T_0, T_1, T_2, T_3...$ a sequence of increasingly strong restrictions on topological spaces. Thus, when we want to study further restrictions, we just add separation axioms.

(It is worth noting that some separation axioms remove the necessity of certain restrictions. For example, if we accept T_1, that any point can be separated from any other point by an open set, we no longer need restriction 4 on closure, $\emptyset$ is closed. **Proof**: A point is in the closure of $\emptyset$ iff all neighborhoods of it have a non-empty intersection with $\emptyset$. By T_1, all points have a neighborhood and since the intersection of any set with $\emptyset$ is empty, are not in $\mathrm{cl}(\emptyset)$. Ergo, $\mathrm{cl}(\emptyset) = \emptyset$ and $\emptyset$ is closed. ■)

I'd like to end this essay by describing where topological spaces stand relative to other spaces. A Hilbert space gives us an idea of angles and distance. A Banach space gives us a notion of magnitude and thus a kind of uniform distance (translation invariant, etc) – it can be generated easily from an inner product space. A metric space gives us a notion of distance and can be created from the norm of a Banach space. Finally, a topological space gives us a notion of adhering – of boundaries, interiors, and so on – and can naturally generated from a metric space: a set is open if it can be created as a union of open balls, an open ball of radius r about a point x being $\{y \in X | d(x, y) < r\}$.



Still, it may seem a bit strange to study the bottom of the latter. The weakness of topological spaces has a very positive side to it, however: everything about topological spaces ripples upwards.